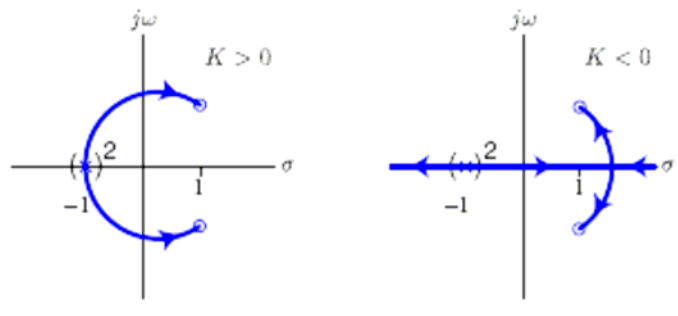


# Signals and Systems

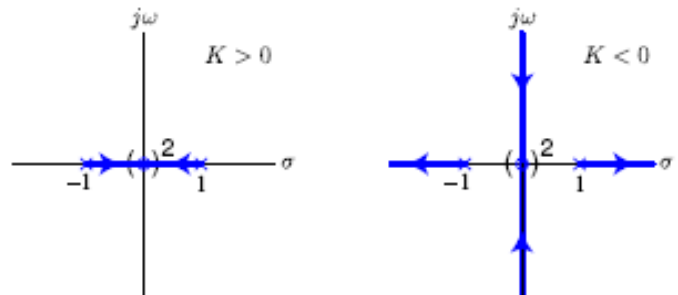
## Solutions to Homework Assignment #8

### Problem 1.

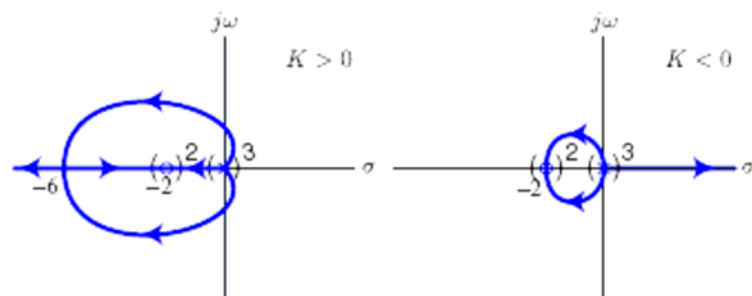
a.



b.



c.



To find the exact breakaway points, note that breakaway/in points are those points such that for some value of  $K$  there are double poles at that point, which is what happens just before the breakaway or just after the breakins. For example, in problem 1c we wish to find the roots to:

$$1 + K \frac{s^2 - 2s + 2}{(s + 1)^2} = 0$$

To simplify, let's write that as  $C(s) = 0$ . Recall that for  $C(s)$  to have double roots at  $s^*$ , two conditions must be satisfied:

- 1 That  $C(s^*) = 0$  (obviously).
- 2 That  $C'(s^*) = 0$ , where  $C'(s)$  is the derivative according to  $s$ .

The second one is so that  $s^*$  is an inflection point, else the roots will not be co-located. In 1c, we obtain:

$$K = -\frac{(s+1)^2}{s^2 - 2s + 2},$$

$$\frac{dK}{ds} = 0 \Rightarrow s = 1.5,$$

which gives the breakaway/breakin point at  $s = 1.5$  for  $K < 0$ , since that point is in the root locus only when  $K < 0$ .

Alternatively, we can use a purely algebraic method. The closed-loop poles are the roots of the following characteristic polynomial in  $s$ :

$$1 + K \frac{s^2 - 2s + 2}{(s+1)^2} = 0 \Rightarrow (s+1)^2 + K(s^2 - 2s + 2) = 0.$$

To find break points where we have a double root on the real axis, we want to use the fact that the characteristic equation can be expressed as “squared”. To see this, we can complete square of the equation above with respect to  $s$ .

$$\begin{aligned} (s+1)^2 + K(s^2 - 2s + 2) &= 0 \\ (1+K)s^2 + (2-2K)s + 1+2K &= 0 \\ s^2 + \frac{2-2K}{1+K}s + \frac{1+2K}{1+K} &= 0 \\ \left(s + \frac{1-K}{1+K}\right)^2 + \frac{1+2K}{1+K} - \left(\frac{1-K}{1+K}\right)^2 &= 0. \end{aligned}$$

Note that we can assume  $1+K \neq 0$  there are two roots at a double pole. Thus, we have

$$\begin{aligned} \frac{1+2K}{1+K} - \left(\frac{1-K}{1+K}\right)^2 &= 0 \\ -K^2 - 5K &= 0 \\ K &= 0, -5. \end{aligned}$$

When  $K = 0$ , we have the  $-\frac{1-K}{1+K} = -\frac{1-(-5)}{1-5} = \frac{3}{2}$  open loop poles as the closed loop poled. When  $K = -5$ , then the closed loop poles are at  $s = -1$  with  $K = 0$  and with  $K = -5$ .

### Problem 2 (O&W 11.57).

Throughout this problem we are interested in analyzing the stability of the system as well as the behavior of  $e(t)$ . The system will be able to track a particular input if  $\lim_{t \rightarrow \infty} e(t) = 0$ . We begin by writing an expression for the closed loop transfer function

$$Q(s) = \frac{Y(s)}{X(s)} = \frac{H_e(s)H_p(s)}{1 + H_e(s)H_p(s)}.$$

We can also write an expression for the Laplace transform of  $e(t)$ ,  $E(s)$ :

$$\begin{aligned} E(s) &= X(s) - Y(s) \\ &= X(s) - E(s)H_e(s)H_p(s) \\ E(s) &= \frac{X(s)}{1 + H_e(s)H_p(s)} \end{aligned}$$

(a) Using the expressions given for  $H_p(s)$  and  $H_e(s)$  we have

$$Q(s) = \frac{\frac{K\alpha}{\alpha+s}}{1 + \frac{K\alpha}{\alpha+s}} = \frac{K\alpha}{s + \alpha(K+1)}$$

Thus, the system will be stable if the pole is in the left-half plane, i.e.  $\alpha(K+1) > 0$ . Given that the system is stable, we can apply the final-value theorem to  $e(t)$  for an input  $x(t) = \delta(t)$ .

$$E(s) = \frac{X(s)}{1 + \frac{K\alpha}{s+\alpha}} = X(s) \frac{s + \alpha}{s + \alpha(1 + K)}$$

Using  $x(t) = \delta(t)$ ,  $X(s) = 1$  and

$$\begin{aligned} E(s) &= \frac{s + \alpha}{s + \alpha(1 + K)} \\ \lim_{t \rightarrow \infty} e(t) &= \lim_{s \rightarrow 0} sE(s) = 0. \end{aligned}$$

For the case when  $x(t) = u(t)$ ,  $X(s) = 1/s$  and

$$\begin{aligned} E(s) &= \frac{s + \alpha}{s + \alpha(1 + K)} \\ \lim_{t \rightarrow \infty} e(t) &= \lim_{s \rightarrow 0} sE(s) = 0. \end{aligned}$$

(b) In order to stabilize the new system, we must choose  $K_1$  and  $K_2$  such that the closed loop system poles are to the left of the  $j\omega$ -axis.

$$\begin{aligned} Q(s) &= \frac{X(s)}{Y(s)} = \frac{H_e(s)H_p(s)}{1 + H_e(s)H_p(s)} \\ &= \frac{\frac{sK_1 + K_2}{s} \frac{\alpha}{\alpha+s}}{1 + \frac{sK_1 + K_2}{s} \frac{\alpha}{\alpha+s}} \\ &= \frac{\alpha K_1 s + \alpha K_2}{s^2 + \alpha(K_1 + 1)s + \alpha K_2} \end{aligned}$$

Thus, the system will be stable if we choose  $K_1$  and  $K_2$  such that  $\alpha(K_1 + 1) > 0$  and  $\alpha K_2 > 0$ . In this case, with  $x(t) = u(t)$ ,  $X(s) = 1/s$ ,  $E(s)$  is given by

$$\begin{aligned} E(s) &= \frac{1}{s} \frac{1}{1 + \frac{sK_1 + K_2}{s} \frac{\alpha}{\alpha+s}} \\ &= \frac{\alpha + s}{s^2 + \alpha(1 + K_1)s + \alpha K_2} \end{aligned}$$

Assuming that the system is stable, we can apply the final-value theorem:

$$\begin{aligned}\lim_{t \rightarrow \infty} e(t) &= \lim_{s \rightarrow 0} sE(s) \\ &= \lim_{s \rightarrow 0} \frac{s^2 + \alpha s}{s^2 + \alpha(1 + K_1)s + \alpha K_2} = 0\end{aligned}$$

Thus, the system can track a step input.

(c) Note that a 3<sup>rd</sup> order polynomial of the form

$$s^3 + \alpha s^2 + \beta s + \gamma$$

will have roots in the left-half plane if  $\alpha, \beta, \gamma > 0$  and  $\beta > \gamma/\alpha$

Using the new  $H_p(s)$  and PI control, we have:

$$\begin{aligned}Q(s) &= \frac{\frac{K_1 s + K_2}{s} \frac{1}{(s-1)^2}}{1 + \frac{K_1 s + K_2}{s} \frac{1}{(s-1)^2}} \\ &= \frac{K_1 s + K_2}{s(s-1)^2 + K_1 s + K_2} \\ &= \frac{K_1 s + K_2}{s^3 - 2s^2 + (K_1 + 1)s + K_2}\end{aligned}$$

In this case it is not possible to stabilize the system because the  $-2s^2$  term. Using PID control we have:

$$\begin{aligned}Q(s) &= \frac{\frac{K_1 s + K_2 + K_3 s^2}{s} \frac{1}{(s-1)^2}}{1 + \frac{K_1 s + K_2 + K_3 s^2}{s} \frac{1}{(s-1)^2}} \\ &= \frac{K_1 s + K_2 + K_3 s^2}{s(s-1)^2 + K_1 s + K_2 + K_3 s^2} \\ &= \frac{K_1 s + K_2 + K_3 s^2}{s^3 + (K_3 - 2)s^2 + (K_1 + 1)s + K_2}\end{aligned}$$

The above system can be stabilized if  $K_1, K_2, K_3$  are chosen such that:

$$\begin{aligned}K_1 &> \frac{K_2}{K_3 - 2} - 1 \\ K_2 &> 0 \\ K_3 &> 2 \\ E(s) &= \frac{1}{s} \frac{1}{1 - \frac{K_1 s + K_2 + K_3 s^2}{s} \frac{1}{(s-1)^2}} \\ &= \frac{(s-1)^2}{s(s-1)^2 - K_1 s - K_2 - K_3 s^2}\end{aligned}$$

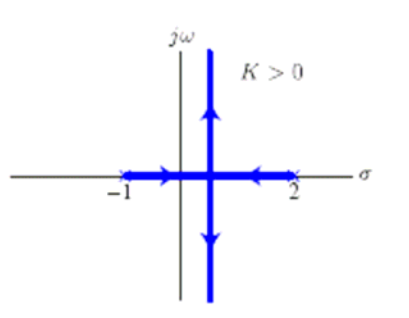
Given that the above system is stable, we apply the final-value theorem:

$$\begin{aligned}\lim_{t \rightarrow \infty} e(t) &= \lim_{s \rightarrow 0} sE(s) \\ &= \lim_{s \rightarrow 0} \frac{s(s-1)^2}{s(s-1)^2 - K_1s - K_2 - K_3s^2} = 0\end{aligned}$$

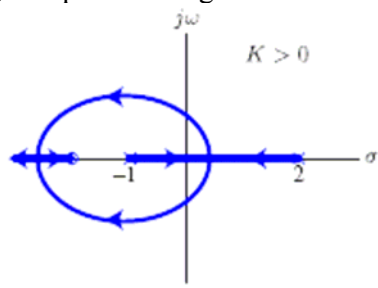
Thus, the system can track a step input.

**Problem 3.**

- a. From the root locus plotted below, it is clear that there will always be at least one closed-loop pole in the right half plane. Therefore, the system cannot be stabilized with proportional feedback.



- b. The desired response corresponds to a double pole on the negative real axis. Proportional plus derivative feedback adds a zero. If the zero is to the left of the leftmost open-loop pole, it is possible to get the desired response, as shown below.



The closed loop response will be given by

$$H(s) = \frac{1}{1 + \frac{K_1 + K_2s}{(s+1)(s-2)}} = \frac{1}{s^2 - s - 2 + K_1 + K_2s}.$$

To get a repeated root at  $s = -\sigma_1$ , the closed loop response must have the form

$$H(s) = \frac{1}{s^2 + 2\sigma_1s + \sigma_1^2}.$$

Therefore

$$K_2 - 1 = 2\sigma_1$$

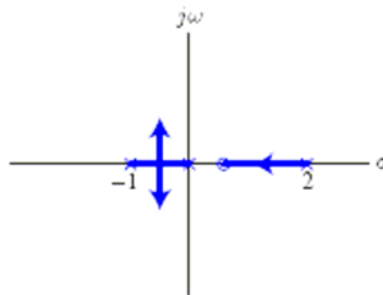
and

$$K_1 - 2 = \sigma_1^2 = \left( \frac{K_2 - 1}{2} \right)^2.$$

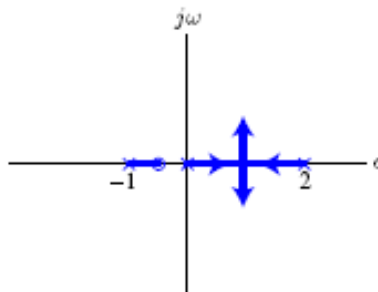
Since  $\sigma_1$  must be greater than zero,  $K_2$  can be any real number bigger than 1. Then

$$K_1 = 2 + \left( \frac{K_2 - 1}{2} \right)^2 \quad \text{and} \quad \sigma_1 = (K_2 - 1)/2.$$

- c. Proportional plus integral feedback adds a pole at zero and a zero at  $s = -K_4/K_3$ . If the zero is in the right half plane, then the system is clearly unstable, as shown in the following root locus.



If the zero is in the left half plane, then the system is clearly oscillatory, as shown in the following root locus.



Thus, it is not possible to locate the zero in a position to get a stable, non-oscillatory response, as would be necessary if the closed-loop impulse response were to be that of a double pole on the real axis.